1/ Math 112 : Introductory Real Analysis

- Summation by parts

Thm (partial summation formula) Given two sequences $\{a_n\}$, $\{b_n\}$, put $A_n := \sum_{k=0}^{n} a_k$ if $n \geq 0$, and put $A_{-1} := 0$.

Then, if $0 \leq p \leq q$, we have

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p$$

proof)

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q} (A_n - A_{n-1}) b_n = \sum_{n=p}^{q} A_n b_n - \sum_{n=p-1}^{q-1} A_n b_{n+1} \qquad \blacksquare$$

The partial summation formula is useful in studying series of the form $\sum a_n b_n$, particularly when $\{b_n\}$ is monotonic.

Thm Suppose

(a) the partial sums A_n of $\sum a_n$ forms a bounded sequence,

(b) $b_0 \geq b_1 \geq b_2 \geq \cdots$,

(c) $\lim_{n\to\infty} b_n = 0$.

Then $\sum a_n b_n$ converges

proof) Choose M such that $|A_n| \leq M$ for all n. Given $\varepsilon > 0$, there is an integer N such that $b_N \leq \frac{\varepsilon}{2M}$. Then, for $N \leq p \leq q$, we have

$$\left|\sum_{n=p}^{q} a_n b_n\right| = \left|\sum_{n=p}^{q} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p\right| \overset{\text{monotonicity of } \{b_n\}}{\leq} M\left|\sum_{n=p}^{q}(b_n - b_{n+1}) + b_q + b_p\right| = 2M b_q \leq 2M b_N \leq \varepsilon \qquad \blacksquare$$

Cor (Alternating series) Suppose

(a) $|c_1| \geq |c_2| \geq \cdots$

(b) $c_{2m-1} \geq 0$, $c_{2m} \leq 0$ $(m = 1, 2, 3, \cdots)$

(c) $\lim_{n\to\infty} c_n = 0$.

Then $\sum c_n$ converges.

proof) Put $a_n = (-1)^{n+1}$, $b_n = |c_n|$ in the previous theorem. ∎

• **Absolute convergence and rearrangements**

Def A series $\sum a_n$ is said to *converge absolutely* if $\sum |a_n|$ converges.

Thm If $\sum a_n$ converges absolutely, then it converges.

proof) This follows easily from $\left|\sum_{k=n}^{m} a_k\right| \leq \sum_{k=n}^{m} |a_k|$ and the Cauchy criterion. ∎

Ex There are series which converge but not absolutely so: for instance

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

Note, the comparison test, the root test, and the ratio test are really tests for absolute convergence.

Summation by parts can sometimes be used to handle non-absolutely convergent series.

<u>Def</u> Let $k : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ be a 1-1 correspondence.

$$n \longmapsto k_n$$

Putting $a_n' = a_{k_n}$ $(n=1,2,3,\cdots)$,

we say that $\sum a_n'$ is a rearrangement of $\sum a_n$.

<u>Thm</u> If $\sum a_n$ is a series (of complex numbers) which converges absolutely, then every rearrangement of $\sum a_n$ converges, and they all converge to the same sum.

proof) Let $\sum a_n'$ be a rearrangement, with partial sums $s_n' := \sum_{k=1}^{n} a_k'$.

Since $\sum |a_n|$ converges, given $\varepsilon > 0$, there exists an integer N such that

$m \geq n \geq N$ implies $\sum_{i=n}^{m} |a_i| \leq \frac{\varepsilon}{2}$.

Choose p such that $\{1, 2, \cdots, N\} \subseteq \{k_1, \cdots, k_p\}$.

Then if $n > p$, $|s_n - s_n'| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Hence $\lim_{n\to\infty} s_n' = \lim_{n\to\infty} s_n$ ∎

(all the $a_1, \cdots, a_N$ cancel)

In fact, the converse is also true in the sense that, if $\sum a_n$ converges non-absolutely, then for any $-\infty \leq \alpha \leq \beta \leq \infty$, there exists a rearrangement $\sum a_n'$ with partial sums s_n' such that $\liminf_{n\to\infty} s_n' = \alpha$ and $\limsup_{n\to\infty} s_n' = \beta$. (See Thm 3.54 in Rudin)

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Thm Let $\sum a_n$ be a series of real numbers which converges but not absolutely.

Suppose $-\infty \leq \alpha \leq \beta \leq \infty$. Then there exists a rearrangement $\sum a_n'$ with partial sums s_n' such that $\liminf_{n\to\infty} s_n' = \alpha$ and $\limsup_{n\to\infty} s_n' = \beta$.

proof) Let $p_n = \dfrac{|a_n| + a_n}{2}$, $q_n = \dfrac{|a_n| - a_n}{2}$ $(n=1,2,3,\cdots)$.

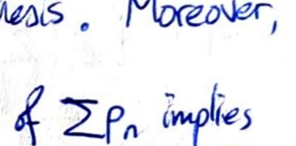
(That is, p_n is the "positive part" and q_n is the "negative part" of a_n.)

Both $\sum p_n$ and $\sum q_n$ must diverge. (They can't both converge because $\sum (p_n + q_n) = \sum |a_n|$ would then converge, contrary to hypothesis. Moreover, since $\sum_{n=1}^{N} a_n = \sum_{n=1}^{N} (p_n - q_n) = \sum_{n=1}^{N} p_n - \sum_{n=1}^{N} q_n$, convergence of $\sum p_n$ implies that of $\sum q_n$ and vice versa.)

Let $P_1, P_2, \cdots$ denote the nonnegative terms of $\sum a_n$ in the order in which they occur, and likewise let $Q_1, Q_2, \cdots$ be the absolute values of the negative terms of $\sum a_n$, also in their original order.

(In otherwords, $\{P_n\}$ and $\{p_n\}$ only differ by the zero terms and likewise for $\{Q_n\}$ and $\{q_n\}$.)

$\sum P_n$ and $\sum Q_n$ are divergent.



We'll create a rearrangement of $\sum a_n$ of the form

$$P_1 + \cdots + P_{m_1} - Q_1 - \cdots - Q_{k_1} + P_{m_1+1} + \cdots + P_{m_2} - Q_{k_1+1} - \cdots - Q_{k_2} + \cdots$$

in the following way: Choose real-valued sequences $\{\alpha_n\}, \{\beta_n\}$ such that $\alpha_n \to \alpha$, $\beta_n \to \beta$, $\alpha_n < \beta_n$ (and $\beta_1 > 0$).

Let m_1, k_1 be smallest integers such that $P_1 + \cdots + P_{m_1} > \beta_1$, $P_1 + \cdots + P_{m_1} - Q_1 - \cdots - Q_{k_1} < \alpha_1$.

Let m_2, k_2 be smallest integers such that $P_1 + \cdots + P_{m_1} - Q_1 - \cdots - Q_{k_1} + P_{m_1+1} + \cdots + P_{m_2} > \beta_2$,

$P_1 + \cdots + P_{m_1} - Q_1 - \cdots - Q_{k_1} + P_{m_1+1} + \cdots + P_{m_2} - Q_{k_1+1} - \cdots - Q_{k_2} < \alpha_2$ etc. This rearrangement satisfies the desired properties. ■